

SOLITONS OF AXION–DILATON GRAVITY

Ioannis Bakas *

Theory Division, CERN, 1211 Geneva 23, Switzerland

and

Department of Physics, University of Patras, 26110 Patras, Greece †

ABSTRACT

We use soliton techniques of the two-dimensional reduced β -function equations to obtain non-trivial string backgrounds from flat space. These solutions are characterized by two integers (n, m) referring to the soliton numbers of the metric and axion-dilaton sectors respectively. We show that the Nappi-Witten universe associated with the $SL(2) \times SU(2)/SO(1, 1) \times U(1)$ CFT coset arises as an $(1, 1)$ soliton in this fashion for certain values of the moduli parameters, while for other values of the soliton moduli we arrive at the $SL(2)/SO(1, 1) \times SO(1, 1)^2$ background. Ordinary 4-dim black-holes arise as 2-dim $(2, 0)$ solitons, while the Euclidean worm-hole background is described as a $(0, 2)$ soliton on flat space. The soliton transformations correspond to specific elements of the string Geroch group. These could be used as starting point for exploring the role of U-dualities in string compactifications to two dimensions.

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*e-mail addresses: BAKAS@SURYA11.CERN.CH, BAKAS@NXTH04.CERN.CH

†Permanent address

1 Introduction

Duality symmetries in string theory arise as discrete remnants of continuous groups of transformations of the lowest order effective theory. These symmetries have received a lot of attention, as they can also provide non-perturbative information about string theory. The most common examples are T and S-dualities, but it has also become clear recently that U-dualities can be successfully used to explore various generalized equivalences among superstrings [1, 2].

Dimensional reduction offers the possibility to intertwine the T and S moduli, and hence construct large groups of solution generating symmetries in three and two dimensions. For example, the reduction from 4 to 3 dimensions gives rise to an $O(2, 2)$ group [3], while the reduction from 4 to 2 dimensions leads to an infinite dimensional group of the lowest order effective theory, the current group $\hat{O}(2, 2)$ [4]. These results can be regarded as straightforward generalization of similar structures found by Geroch in the space of solutions of vacuum Einstein equations with one or two commuting isometries [5], but now they also include apart from the metric $G_{\mu\nu}$ the antisymmetric tensor field $B_{\mu\nu}$ and the dilaton Φ . The coset space structure of the scalar fields in various dimensionally reduced supergravity theories was known before (see for instance [6, 7, 8] and references therein). More recent is the realization that T and S dualities are embedded in the corresponding continuous hidden symmetry groups. In a heterotic string context it means that the reduction from 10 to 3 dimensions gives rise to an $O(8, 24)$ group [7, 9], while the reduction from 10 to 2 dimensions leads to $\hat{O}(8, 24)$ [10, 11]. It is then natural to expect that the 2-dim sector of string theory will be quite rich in symmetry, having as U-duality an appropriately chosen discrete subgroup of the underlying string Geroch current group. Up to this day, however, very little progress has been made in this particular direction, since proving the conjectured U-dualities and understanding their action on the full spectrum of superstring models based on these effectively 2-dim backgrounds turns into a difficult problem.

In this paper we consider string models with target space $M_4 \times K$, where M_4 is a 4-dim spacetime with signature $-+++$ and K is some internal space, which is usually represented by a conformal field theory (CFT), so that the total central charge is critical. We focus on cosmological backgrounds M_4 with non-trivial $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ that arise as solutions of the lowest order effective theory,

$$S_{eff} = \int_{M_4} d^4X \sqrt{-\det G} \left(R - 2(\nabla_\mu \Phi)^2 - \frac{1}{12} e^{-4\Phi} H_{\mu\nu\rho}^2 \right). \quad (1.1)$$

Here the theory is defined directly in the Einstein frame, which is related to the σ -model frame by $G_{\mu\nu}^{(\sigma)} = e^{2\Phi} G_{\mu\nu}$, and the effective cosmological constant is taken zero. It will be convenient for later use to trade $B_{\mu\nu}$ with the axion field b , which is consistently defined in the Einstein frame as follows:

$$\partial_\mu b = \frac{1}{6} e^{-4\Phi} \sqrt{-\det G} \epsilon_\mu^{\nu\rho\sigma} H_{\nu\rho\sigma}. \quad (1.2)$$

$H_{\mu\nu\rho}$ is the field strength of $B_{\mu\nu}$ and $\epsilon_{0123} = 1$. In M_4 with signature $-+++$ we may further define the complex conjugate fields $S_{\pm} = b \pm ie^{-2\Phi}$, which provide the natural variables of the S-moduli. Later we will also consider string backgrounds with Euclidean signature $++++$.

There is a very limited number of exact CFT backgrounds, which to lowest order in α' provide solutions of (1.1). The most characteristic examples of this type are the two pairs of WZW coset models

$$\frac{SL(2) \times SU(2)}{SO(1,1) \times U(1)} , \quad \frac{SL(2)}{SO(1,1)} \times SO(1,1)^2 , \quad (1.3)$$

$$\frac{SL(2)}{SO(1,1)} \times \frac{SU(2)}{U(1)} , \quad SL(2) \times SO(1,1) . \quad (1.4)$$

The first model in (1.3) depends on a free parameter that defines the gauging of the coset, and it is particularly interesting in string cosmology as it describes a closed inhomogeneous expanding and recollapsing universe [12] (see also [13] for some earlier ideas). The other three models are the Lorentzian counterparts obtained by analytic continuation of the $D = 4$, $\hat{c} = 4$, $N = 4$ superconformal backgrounds $C^{(4)} = SU(2)/U(1) \times U(1)^2$, $\Delta^{(4)} = SL(2)/U(1) \times SU(2)/U(1)$ and $W^{(4)} = SU(2) \times U(1)$ (the throat of a worm-hole) respectively with appropriately chosen background charges [14, 15, 16]. All these models exhibit two commuting Killing isometries. It has been established with the aid of $O(2,2)$ transformations that the first model in (1.3) is related to the first model in (1.4) [17], and similarly the other two models of the series are T-dual to each other [15, 18].

Our contribution is to connect the two gravitational backgrounds associated with the Nappi-Witten universe $SL(2) \times SU(2)/SO(1,1) \times U(1)$ and $SL(2)/SO(1,1) \times SO(1,1)^2$ to the trivial flat space background $F^{(4)}$ with zero $B_{\mu\nu}$ and Φ , by considering a specially chosen six dimensional moduli space of backgrounds within the entire set of solutions of (1.1) with two commuting Killing isometries. This is technically achieved by performing first the 2-dim reduction of the effective theory (1.1), and then employing solitonic constructions that are available for the resulting integrable system of equations (both for the metric and the axion-dilaton sectors). As it turns out, the simplest $(1,1)$ soliton configuration on $F^{(4)}$, or more precisely its T-dual face, will be sufficient to describe the semiclassical backgrounds of these two coset models as 2-dim solitons for appropriate choices of the six moduli parameters. The solitonic dressing of (the dual of) $F^{(4)}$ in this paper is analogous to the solitonic dressing of Kasner type metrics that were studied by Belinski and Sakharov in the context of general relativity many years ago [19]. In the context of pure gravity these authors gave a very interesting derivation of 4-dim black holes as 2-dim double soliton solutions on flat space. Further work has also appeared in the literature, which describes the physically very interesting situation of two colliding gravitational plane waves in terms of 2-dim solitons [20]. Given the wide applicability of these methods, it is also natural to consider the explicit form of solitons in axion-dilaton gravity and attempt a reinterpretation of known solutions, in particular those that correspond to exact CFT backgrounds, in this context.

It is interesting to note that the soliton dressing of a given configuration corresponds to a specific choice of finite group element of the Geroch group (see for instance [21]), and hence in the string context we have found the way to generate the two exact CFT backgrounds (1.3) from flat space by U-duality (viewed as a continuous group of transformations at this point). In a sense one may then say that the dualities create a universe, the Nappi-Witten cosmological solution in this case. We will also consider Euclidean string backgrounds and show for example that the worm-hole solution, which is an axionic instanton of the 10-dim heterotic theory, arises as a 2-dim $(0, 2)$ soliton on flat space. Certainly, many more connections can be made between different string backgrounds using the inverse scattering method of the 2-dim reduced sector, and we hope to return to them in the near future.

In section 2 we briefly discuss the dimensionally reduced string background equations and outline the construction of soliton solutions using the integrability of the resulting 2-dim σ -models. In section 3 we construct the most general $(1, 1)$ soliton solution on a T-dual face of Minkowski space and determine the choice of moduli parameters that correspond to the CFT backgrounds $SL(2) \times SU(2)/SO(1, 1) \times U(1)$ and $SL(2)/SO(1, 1) \times SO(1, 1)^2$. In section 4 we describe the ordinary 4-dim black-holes as 2-dim $(2, 0)$ solitons on flat space, where the soliton moduli correspond to the mass, rotation and NUT parameters of the most general stationary axisymmetric solution. In section 5 the Euclidean worm-hole background is interpreted as a 2-dim $(0, 2)$ soliton in the same context. Section 6 contains our conclusions and some directions for further work on the subject. We argue that the present results could be most importantly used as starting point for exploring the role of U-dualities in compactifications of string theory to two dimensions.

2 The reduced theory and its solitons

The effective theory (1.1) describes the coupling of an ordinary $SL(2)/U(1)$ σ -model to 4-dim gravity, which is manifest in the axion-dilaton formulation using the field variables $S_{\pm} = b \pm ie^{-2\Phi}$. This axion-dilaton σ -model has Lorentzian signature if M_4 is Euclidean, but in the case of interest here, where M_4 is Lorentzian, the $SL(2)/U(1)$ σ -model is Euclidean. Hence, it is convenient to parametrize the axion-dilaton sector by the symmetric matrix

$$\lambda = e^{2\Phi} \begin{pmatrix} 1 & b \\ b & b^2 + e^{-4\Phi} \end{pmatrix} \quad (2.1)$$

so that $\det \lambda = 1$.

Following [4] (and references therein) we consider gravitational string backgrounds of cosmological type with two commuting Killing isometries, so that the target space metric

is restricted by the ansatz

$$ds^2 = f(X^0, X^1) \left(-(dX^0)^2 + (dX^1)^2 \right) + g_{AB}(X^0, X^1) dX^A dX^B ; \quad A, B = 2, 3, \quad (2.2)$$

and also $b(X^0, X^1)$, $\Phi(X^0, X^1)$. For notational convenience we introduce light-cone coordinates

$$\eta = \frac{1}{2}(X^0 - X^1) , \quad \xi = \frac{1}{2}(X^0 + X^1) . \quad (2.3)$$

Then, using the defining relations (1.2) this ansatz amounts to choosing special backgrounds with only $B_{23} \neq 0$, in which case the axion equations simplify to

$$\partial_\xi b = \frac{e^{-4\Phi}}{\sqrt{\det g}} \partial_\xi B_{23} , \quad \partial_\eta b = -\frac{e^{-4\Phi}}{\sqrt{\det g}} \partial_\eta B_{23} . \quad (2.4)$$

This class of backgrounds will be quite sufficient for the present purposes of our work.

The reduced string background equations that follow from (1.1) read as follows in the Einstein frame:

$$\partial_\eta(\sqrt{\det g} g^{-1} \partial_\xi g) + \partial_\xi(\sqrt{\det g} g^{-1} \partial_\eta g) = 0 , \quad (2.5)$$

$$\partial_\eta(\sqrt{\det g} \lambda^{-1} \partial_\xi \lambda) + \partial_\xi(\sqrt{\det g} \lambda^{-1} \partial_\eta \lambda) = 0 . \quad (2.6)$$

They are two essentially decoupled $SL(2)/U(1)$ 2-dim σ -models of Ernst type (both having Euclidean signature), one for the metric sector g and one for the axion-dilaton λ . Since $\sqrt{\det g}$ satisfies the 2-dim wave equation $\partial_\eta \partial_\xi \sqrt{\det g} = 0$, we may choose without loss of generality

$$X^0 = \sqrt{\det g} \equiv \alpha , \quad X^1 = \beta \quad (2.7)$$

for the corresponding pair of its conjugate solutions. From now on we assume the special choice of coordinates (2.7), using $\alpha = \sqrt{\det g}$ and β instead of X^0 and X^1 in (2.2), and reserve the notation X^0, X^1 for more general coordinate systems. Sometimes we will also denote the remaining two coordinates by z and w instead of X^2 and X^3 , in order to make more uniform our presentation in the following sections.

We recall that the differential equations for the conformal factor f are linear of first order,

$$\partial_\xi(\log f) = -\frac{1}{\alpha} + \frac{\alpha}{4} \text{Tr} \left((g^{-1} \partial_\xi g)^2 + (\lambda^{-1} \partial_\xi \lambda)^2 \right) , \quad (2.8)$$

$$\partial_\eta(\log f) = -\frac{1}{\alpha} + \frac{\alpha}{4} \text{Tr} \left((g^{-1} \partial_\eta g)^2 + (\lambda^{-1} \partial_\eta \lambda)^2 \right) \quad (2.9)$$

and so, once a solution (g, λ) of the two Ernst σ -models is known, f can be simply determined integrating by quadratures. All these calculations are performed in the Einstein frame, where the decoupling of (2.5) and (2.6) takes place, but the results can be easily translated in the σ -model frame of string theory.

The non-linear σ -models of the 2-dim reduced theory are known to be integrable, and it is precisely this property that is responsible for having an infinite dimensional symmetry group, the (string) Geroch group, acting on the space of classical solutions. Because of integrability the 2-dim σ -models admit soliton solutions, which can be constructed

explicitly on any given background that acts as a seed for the solitons. On any string background we may actually construct a whole series of solitonic excitations (n, m) , where n and m denote the soliton numbers of the g and λ sectors respectively. Here we briefly review the essential ingredients of the soliton technique for the Ernst σ -model using only the metric sector of the theory, but the construction is exactly the same for the axion-dilaton sector since $\sqrt{\det g} \lambda$ satisfies the same equation (2.6) as λ . The new solutions that arise in this fashion are solitons in the 2-dim sense, and although 4-dim string backgrounds can be reconstructed from the (n, m) data, the resulting configurations are not necessarily solitons of the 4-dim world saturating the Bogomol'ny bound, and thus they are generically quantum mechanically unstable. Next, we revisit the soliton framework of Belinski-Sakharov [19], because their technique is not widely known to string theorists.

Consider the following linear system of (2×2) -matrix differential equations:

$$D_1 \Psi = \frac{A}{l - \alpha} \Psi, \quad D_2 \Psi = \frac{B}{l + \alpha} \Psi, \quad (2.10)$$

where $\Psi(\eta, \xi; l)$ is a complex matrix function depending on a spectral parameter l that takes values in the whole complex plane, and

$$A = -\alpha \partial_\xi g g^{-1}, \quad B = \alpha \partial_\eta g g^{-1}. \quad (2.11)$$

Also, the differential operators are

$$D_1 = \partial_\xi - 2 \frac{l}{l - \alpha} \partial_l, \quad D_2 = \partial_\eta + 2 \frac{l}{l + \alpha} \partial_l \quad (2.12)$$

and clearly they commute, $[D_1, D_2] = 0$. The system (2.10) is compatible provided that g satisfies the Ernst equation (2.5) for $\sqrt{\det g} = \alpha$. We actually assume

$$\Psi(\eta, \xi; l = 0) = g(\eta, \xi), \quad (2.13)$$

and so Ψ can be regarded as a suitable generalization of g with spectral parameter.

Let $g_0(\eta, \xi)$ be a known solution and let $\Psi_0(\eta, \xi; l)$ be the corresponding solution of the linear system (2.10). If we assume that other solutions g exist such that

$$\Psi(l) = \chi(l) \Psi_0(l), \quad (2.14)$$

then $\chi(\eta, \xi; l)$ has to satisfy the system of equations

$$D_1 \chi = \frac{1}{l - \alpha} (A \chi - \chi A_0), \quad D_2 \chi = \frac{1}{l + \alpha} (B \chi - \chi B_0), \quad (2.15)$$

where A_0, B_0 are the currents (2.11) of a seed metric g_0 . If we manage to find an appropriate $\chi(l)$, then according to (2.13) and (2.14), a new solution will be obtained

$$g(\eta, \xi) = \chi(l = 0) g_0. \quad (2.16)$$

There are a few technical assumptions on χ , namely the reality condition on the real l -line, $\bar{\chi}(\bar{l}) = \chi(l)$, and $\chi(\infty) = 1$.

The n -soliton excitations of a given seed background g_0 are very special in that χ has a simple pole structure in the complex l -plane

$$\chi(\eta, \xi; l) = 1 + \sum_{k=1}^n \frac{R_k(\eta, \xi)}{l - \mu_k(\eta, \xi)} . \quad (2.17)$$

The residue and pole functions can be determined substituting (2.17) in (2.15) and start comparing the pole structure on the left and right hand sides. The details are rather lengthy and we skip them here. We only give the final result that will be used later for explicit computations. The poles are roots of the algebraic equation

$$\mu_k^2 + 2(\beta - C_0^{(k)})\mu_k + \alpha^2 = 0 , \quad (2.18)$$

where $C_0^{(k)}$ are arbitrary numerical constants (moduli), and μ_k satisfy the differential equations in η, ξ ,

$$\partial_\eta \mu_k = \frac{2\mu_k}{\alpha + \mu_k} , \quad \partial_\xi \mu_k = \frac{2\mu_k}{\alpha - \mu_k} . \quad (2.19)$$

The residue matrices R_k are degenerate having the component form

$$(R_k)_{AB} = N_A^{(k)} M_B^{(k)} , \quad (2.20)$$

where the 2-component vector $M^{(k)}$ is given using the inverse of Ψ_0 at $l = \mu_k$,

$$M_B^{(k)} = \sum_A C_A^{(k)} \Psi_0^{-1}(\eta, \xi; l = \mu_k)_{AB} . \quad (2.21)$$

$C^{(k)}$ is an arbitrary constant 2-component vector, which together with $C_0^{(k)}$ provide all the moduli parameters of the general solitonic excitation of g_0 . The other 2-component vectors $N^{(k)}$ have more complicated form and can not be explicitly written with the same ease. $N^{(k)}$ are the solution vectors of the n -th order linear system of algebraic equations

$$\sum_{l=1}^n \Gamma_{kl} N_A^{(l)} = \frac{1}{\mu_k} \sum_B M_B^{(k)} (g_0)_{AB} , \quad (2.22)$$

where the $n \times n$ matrix Γ was determined by Belinski and Sakharov

$$\Gamma_{kl} = \frac{1}{\mu_k \mu_l - \alpha^2} \sum_{A,B} M_A^{(k)} (g_0)_{AB} M_B^{(l)} . \quad (2.23)$$

Therefore, putting it all together we arrive at a concrete expression for the n -soliton excitation of g_0 , namely

$$g(\eta, \xi) = \left(1 - \sum_{k=1}^n \frac{R_k}{\mu_k} \right) g_0 . \quad (2.24)$$

A final issue is the overall normalization of the dressing matrix χ . Using this last equation we find $\det g = \alpha^{2n+2} \mu_1^{-2} \mu_2^{-2} \cdots \mu_n^{-2}$, which differs from $\det g_0 = \alpha^2$. Agreement is

achieved by scaling χ with $\mu_1\mu_2\cdots\mu_n/\alpha^n$, and this is what we will assume from now on. The properly normalized n -soliton dressing matrices of g_0 ,

$$\chi(l=0) = \frac{\mu_1\mu_2\cdots\mu_n}{\alpha^n} \left(1 - \sum_{k=1}^n \frac{R_k}{\mu_k}\right), \quad (2.25)$$

define specific group elements of the Geroch group. The normalization (2.25) is introduced to achieve consistency with the standard formulation of Geroch transformations that preserve $\det g$.

Summarizing, if we apply this procedure to any given string background g_0 , λ_0 (more precisely $\alpha\lambda_0$ to be in exact analogy for both sectors), we will obtain a generic (n, m) solitonic excitation with $3n+3m$ continuous moduli. One of the difficulties to implement this construction in practice, apart from the problem of inverting the corresponding matrices Γ for large n and m , is to find the explicit solution Ψ_0 of (2.10) (and its λ -counterpart) when an arbitrary background is used as seed. For this reason we will start from very simple seed solutions, knowing Ψ_0 , and use the soliton technique to construct (and hence reinterpret in this context) the more complicated solutions that exist in the literature.

The Geroch group of the metric sector is the loop group $\hat{SL}(2)$, and when both sectors are taken into account the Geroch group becomes $\hat{SL}(2) \times \hat{SL}(2) = \hat{O}(2, 2)$. It is known in this case how to obtain the entire algebra by successive intertwining of continuous T and S transformations [4]. We briefly mention here that 4-dim backgrounds with two commuting isometries exhibit the obvious $O(2, 2)$ group of transformations on the space of solutions. These transformations are non-locally realized in the axion-dilaton formulation of the theory and their generators are embedded in the algebra of the string Geroch group $\hat{O}(2, 2)$ as follows: we use the zero mode subalgebra of the g - $\hat{SL}(2)$, say T_+^0 , T_-^0 , T_0^0 , and the non-locally realized $SL(2)$ subalgebra of the λ - $\hat{SL}(2)$, \tilde{T}_+^{-1} , \tilde{T}_0^0 , \tilde{T}_-^1 that includes the ± 1 modes. The continuous analogue of the S-duality $SL(2)$ transformations are locally realized in the axion-dilaton formulation and correspond to the zero mode generators \tilde{T}_+^0 , \tilde{T}_-^0 , \tilde{T}_0^0 of the λ - $\hat{SL}(2)$. Hence, by intertwining $O(2, 2)$ with S we can generate after an infinite number of steps the entire $\hat{SL}(2)$ algebra of the axion-dilaton sector. To generate the other $\hat{SL}(2)$ we interchange the field variables $g \leftrightarrow \alpha\lambda$ and perform the same intertwining procedure. The exchange of the two sectors is a legitimate operation in this case because both σ -models have Euclidean signature, and this is also a Z_2 symmetry of the 2-dim reduced string background equations, leaving f unaffected.

The n -soliton matrices (2.25) could also be described in terms of specific elements of the infinite dimensional group of Geroch transformations obtained by other approaches [21]. We will not attempt here to decompose them in terms of more fundamental operations associated to successive intertwining of T and S transformations, but we note as an important property their commutativity in the following sense: an $(n + n')$ -soliton can either be constructed directly from a seed background or it can be viewed as an n' -solitonic excitation of the n -soliton, and similarly for $n \leftrightarrow n'$. Since there is a systematic understanding of the group elements of soliton dressing, we think that is worth exploring

further the precise meaning and the consequences of U-duality in this particular sector of string theory. Of course much work remains to be done in this direction.

3 (1, 1) solitons and CFT backgrounds

According to the general framework of the previous section we may compute the simplest 1-soliton solution of the Ernst σ -model, say (2.5), using as seed metric

$$g_0 = \begin{pmatrix} \alpha^{2s_1} & 0 \\ 0 & \alpha^{2s_2} \end{pmatrix}; \quad s_1 + s_2 = 1. \quad (3.1)$$

In a purely gravitational context this choice of the seed metric corresponds to a Kasner cosmological background. There are two special cases in this family, namely

$$s_1 = 0, \quad s_2 = 1; \quad s_1 = s_2 = \frac{1}{2}, \quad (3.2)$$

which correspond to flat space (in polar coordinates) and an isotropic universe respectively. Using (2.10) we may determine Ψ_0 for this background,

$$\Psi_0(l) = \begin{pmatrix} (l^2 + 2\beta l + \alpha^2)^{s_1} & 0 \\ 0 & (l^2 + 2\beta l + \alpha^2)^{s_2} \end{pmatrix}. \quad (3.3)$$

We see clearly that $\Psi_0(l=0) = g_0$ as required on general grounds (2.13).

We will first derive the general form of the 1-soliton solution, and then make various specializations according to the connections we would like to make later with 4-dim CFT backgrounds. The 1-soliton background on g_0 (3.1) is obtained using only one pole located at

$$\mu = \mu_{\pm} = C_0 - \beta \pm \sqrt{(C_0 - \beta)^2 - \alpha^2}. \quad (3.4)$$

This pole is positioned on the real axis of the complex l -plane provided that

$$(C_0 - \beta)^2 \geq \alpha^2. \quad (3.5)$$

Otherwise we will be forced to consider more complicated double soliton solutions, since complex poles always come in pairs. Then, the physical 1-soliton matrix (after normalization with μ/α) reads

$$g_1 = \frac{\alpha}{\mu A} \begin{pmatrix} C_2^2(2C_0\mu)^{2s_1}\mu^2 + C_1^2(2C_0\mu)^{2s_2}\alpha^{4s_1} & 2C_0C_1C_2\mu(\alpha^2 - \mu^2) \\ 2C_0C_1C_2\mu(\alpha^2 - \mu^2) & C_1^2(2C_0\mu)^{2s_2}\mu^2 + C_2^2(2C_0\mu)^{2s_1}\alpha^{4s_2} \end{pmatrix}, \quad (3.6)$$

where

$$A = C_1^2 (2C_0\mu)^{2s_2} \alpha^{2s_1} + C_2^2 (2C_0\mu)^{2s_1} \alpha^{2s_2} . \quad (3.7)$$

If we were to apply the same construction to the axion-dilaton Ernst σ -model (2.6), we should have scaled α in front of the seed matrix (3.1), since g_0 behaves the same way as $\sqrt{\det g} \lambda_0 = \alpha \lambda_0$. Let us begin with a background having

$$e^{-2\Phi_0} = \alpha^{2s'_2-1} ; \quad b_0 = 0 \quad (3.8)$$

with $s'_1 + s'_2 = 1$ as well. Scaling out α from the general form of the 1-soliton solution, we find that the new axion-dilaton system is given by the configuration

$$e^{-2\Phi_1} = \mu \frac{C_1'^2 (2C_0'\mu')^{2s'_2} \alpha^{2s'_1} + C_2'^2 (2C_0'\mu')^{2s'_1} \alpha^{2s'_2}}{C_1'^2 (2C_0'\mu')^{2s'_2} \alpha^{4s'_1} + C_2'^2 (2C_0'\mu')^{2s'_1} \mu'^2} , \quad (3.9)$$

$$b_1 = \frac{2C_0' C_1' C_2' \mu' (\alpha^2 - \mu'^2)}{C_1'^2 (2C_0'\mu')^{2s'_2} \alpha^{4s'_1} + C_2'^2 (2C_0'\mu')^{2s'_1} \mu'^2} , \quad (3.10)$$

where the primes are used to distinguish the parameters of the axion-dilaton system from those of the metric moduli.

Suppose now we are combining both sectors to construct the (1, 1) soliton starting from the following solution of the string background equations in the Einstein frame:

$$ds^2 = -d\alpha^2 + d\beta^2 + \alpha(dz^2 + dw^2) , \quad (3.11)$$

$$b_0 = 0 , \quad e^{-2\Phi_0} = \alpha . \quad (3.12)$$

This particular choice of the seed background is very special in that is T-dual to $F^{(4)}$, i.e., the flat space metric with zero dilaton and antisymmetric tensor fields. To see this we translate (3.11) in the σ -model frame and perform T-duality with respect to the Killing coordinate β , which yields the purely gravitational background

$$ds^2 = -\frac{1}{\alpha} d\alpha^2 + \alpha d\beta^2 + dz^2 + dw^2 . \quad (3.13)$$

Introducing coordinates

$$x = 2\sqrt{\alpha} \cosh \frac{\beta}{2} , \quad y = 2\sqrt{\alpha} \sinh \frac{\beta}{2} , \quad (3.14)$$

the metric (3.13) assumes the flat space form $ds^2 = -dx^2 + dy^2 + dz^2 + dw^2$. Actually it is immediately recognized that (3.14) is a Rindler transformation of the 2-dim Minkowski space (x, y) with $(\log \alpha, \beta)$ providing the corresponding pair of Rindler coordinates. Hence, we start from the 2-dim Rindler wedge times a flat 2-dim Euclidean space parametrized by the other two coordinates (z, w) , and use its T-dual face as seed string background.

We notice that our ansatz (3.11), (3.12) for the seed background imply the following choice of the Kasner type parameters for the two sectors,

$$g : s_1 = s_2 = \frac{1}{2} , \quad (3.15)$$

$$\lambda : s'_1 = 0 , \quad s'_2 = 1 . \quad (3.16)$$

Then the resulting $(1, 1)$ soliton simplifies considerably and in the Einstein frame is given by

$$g_1 = \frac{1}{C_1^2 + C_2^2} \begin{pmatrix} C_1^2 \frac{\alpha^2}{\mu} + C_2^2 \mu & C_1 C_2 \left(\frac{\alpha^2}{\mu} - \mu \right) \\ C_1 C_2 \left(\frac{\alpha^2}{\mu} - \mu \right) & C_1^2 \mu + C_2^2 \frac{\alpha^2}{\mu} \end{pmatrix}, \quad (3.17)$$

$$e^{-2\Phi_1} = \frac{1}{4C_0'^2 C_1'^2 + C_2'^2} \left(4C_0'^2 C_1'^2 \mu' + C_2'^2 \frac{\alpha^2}{\mu'} \right), \quad (3.18)$$

$$b_1 = \frac{2C_0' C_1' C_2'}{4C_0'^2 C_1'^2 + C_2'^2} \left(\frac{\alpha^2}{\mu'} - \mu' \right). \quad (3.19)$$

As for the conformal factor f , which follows by integration of (2.8), (2.9), we find after some lengthy computation the result

$$f_1 = \frac{\mu(4C_0'^2 C_1'^2 \mu'^2 + C_2'^2 \alpha^2)}{(\alpha^2 - \mu^2)(\alpha^2 - \mu'^2)} \quad (3.20)$$

up to an overall numerical factor, whereas $f_0 = 1$ by inspecting (3.11).

There is an ambiguity to choose μ_+ or μ_- in (3.17)–(3.19), but since

$$\mu_{\pm} = \frac{\alpha^2}{\mu_{\mp}}, \quad (3.21)$$

the two choices yield the same result provided that in the metric soliton moduli space $(C_1, C_2) \rightarrow (-C_2, C_1)$. Similarly in the axion-dilaton sector the two choices μ'_{\pm} are equivalent provided that $(C'_1, C'_2) \rightarrow (-C'_2/2C'_0, 2C'_0 C'_1)$. Hence in the following we may choose without loss of generality

$$\mu = C_0 - \beta + \sqrt{(C_0 - \beta)^2 - \alpha^2}, \quad \mu' = C'_0 - \beta + \sqrt{(C'_0 - \beta)^2 - \alpha^2}. \quad (3.22)$$

Next, we show how to obtain the string backgrounds associated to the two coset models $SL(2) \times SU(2)/SO(1, 1) \times U(1)$ and $SL(2)/SO(1, 1) \times SO(1, 1)^2$ by making appropriate choices of the moduli parameters in the 6-dim space of solutions we have obtained.

(i) Nappi-Witten universe :

In the g -sector of the general $(1, 1)$ soliton solution we choose

$$C_1 = 0, \quad (3.23)$$

which gives rise to a diagonal metric with components

$$g_{zz} = C_0 - \beta + \sqrt{(C_0 - \beta)^2 - \alpha^2}, \quad g_{ww} = C_0 - \beta - \sqrt{(C_0 - \beta)^2 - \alpha^2} \quad (3.24)$$

independent of C_2 . For the axion-dilaton sector we set

$$C'_0 = 1, \quad \frac{C'_2}{2C'_1} = \frac{\sin \theta - 1}{\cos \theta}, \quad (3.25)$$

where θ is an arbitrary numerical constant. Hence, choosing μ'_+ we also fix

$$e^{-2\Phi} = 1 - \beta + \sin \theta \sqrt{(1 - \beta)^2 - \alpha^2} , \quad (3.26)$$

$$b = \cos \theta \sqrt{(1 - \beta)^2 - \alpha^2} , \quad (3.27)$$

while the conformal factor in the Einstein frame is determined according to (3.20).

We claim that this solution corresponds to the cosmological background found by Nappi and Witten while considering the $SL(2) \times SU(2)/SO(1,1) \times U(1)$ CFT coset. In this regard, the numerical parameter θ that was introduced in (3.25) will be shown to describe the arbitrariness in the gauging of this coset. For this purpose we also choose

$$C_0 = -1 , \quad (3.28)$$

thus describing the same θ -dependent string background for any point in the soliton moduli space that is restricted by (3.23), (3.25) and (3.28).

The construction is rather formal up to now, while making various seemingly unjustified choices of the free parameters. At this point we introduce coordinates X^0, X^1 in terms of α, β given by

$$\alpha = \sin 2X^0 \sin 2X^1 , \quad \beta = \cos 2X^0 \cos 2X^1 , \quad (3.29)$$

thus also restricting the range of α and β as X^0 and X^1 range from 0 to $\pi/2$. This is a good choice because

$$e^{-2\Phi} = 1 - \cos 2X^0 \cos 2X^1 + \sin \theta (\cos 2X^0 - \cos 2X^1) \quad (3.30)$$

is manifestly real and positive, as should be expected for an honest dilaton field. For the axion we find

$$b = \cos \theta (\cos 2X^0 - \cos 2X^1) . \quad (3.31)$$

At first sight it seems that these choices are not good for the metric sector (3.24), since

$$g_{zz} = -4\sin^2 X^0 \sin^2 X^1 , \quad g_{ww} = -4\cos^2 X^0 \cos^2 X^1 , \quad (3.32)$$

and the signature turns out to be $--$ instead of $++$. Recall, however, the way we have obtained the physical metric in the soliton construction of section 2. There, we had to scale $\chi(l=0)$ accordingly so that $\det g = \det g_0 = \alpha^2$. The scaling was μ/α for the 1-soliton, but equally well we could have taken $-\mu/\alpha$. The latter choice renders the signature of g physical, i.e. $++$, and there is no contradiction.

To make exact contact with the Nappi-Witten cosmological background we introduce coordinates X^2 and X^3 by scaling

$$w = \sqrt{\frac{1 + \sin \theta}{2}} X^2 , \quad z = \sqrt{\frac{1 - \sin \theta}{2}} X^3 \quad (3.33)$$

and compute the full metric in the σ -model frame. The final result reads

$$ds_{(\sigma)}^2 = -(dX^0)^2 + (dX^1)^2 + \frac{2}{1 - \cos 2X^0 \cos 2X^1 + \sin \theta (\cos 2X^0 - \cos 2X^1)} \cdot \\ \left((1 + \sin \theta) \cos^2 X^0 \cos^2 X^1 (dX^2)^2 + (1 - \sin \theta) \sin^2 X^0 \sin^2 X^1 (dX^3)^2 \right). \quad (3.34)$$

We also compute the antisymmetric tensor field from the axion and find that all its components are zero apart from

$$B_{23} = \frac{1}{2} \frac{\cos 2X^0 - \cos 2X^1 + \sin \theta (1 - \cos 2X^0 \cos 2X^1)}{1 - \cos 2X^0 \cos 2X^1 + \sin \theta (\cos 2X^0 - \cos 2X^1)}. \quad (3.35)$$

This is precisely the result that was obtained in the semi-classical limit of the $SL(2) \times SU(2)/SO(1,1) \times U(1)$ coset model having an arbitrary parameter θ that specifies the gauging [12].

So, according to this, the Nappi-Witten universe can be created from flat space starting from (a suitably restricted part of) the Rindler wedge, performing a T-duality transformation and then an $(1,1)$ soliton dressing. Consequently, our procedure completely determines the group element of the string Geroch group $\hat{O}(2,2)$ that connects classically the two backgrounds. The Nappi-Witten background describes a closed expanding and recontracting universe as X^0 varies from 0 (big bang) to $\pi/2$ (big crunch). These two authors performed an in depth analysis of the model noting that for

$$\frac{1 - \sin \theta}{\cos \theta} = \text{rational number} \quad (3.36)$$

$X^0 = 0$ or $\pi/2$ are orbifold singularities. Also, away from the special values $X^1 = 0$ or $\pi/2$ respectively, these are singularities in the causal structure of space-time rather than curvature singularities.

This cosmological solution is positioned in the entire moduli space of $(1,1)$ solitons as follows: consider the 3-dim subspace with axis labelled by C_2 , C'_1 and C'_2 , while keeping the other coordinates fixed to their chosen values $C_1 = 0$, $C'_0 = 1 = -C_0$; if we draw all 2-dim planes having the C_2 -line as common axis, then every point on each such plane will correspond to the same solution, while rotating planes change θ . In this description the criterion (3.35) for having orbifold singularities is equivalent to considering rational values for the slope of the solution plane, which is given by C'_2/C'_1 according to (3.25).

Concluding we mention that the points of the moduli space with the same restrictions as before, but with $C'_2 = 0$, yield (a suitable analytic continuation of) the background $SL(2)/SO(1,1) \times SU(2)/U(1)$. Using the parametrization (3.29) it follows from our general expression that the axion field is zero, the dilaton field is

$$e^{-2\Phi} = \cos^2 X^0 \sin^2 X^1, \quad (3.37)$$

and the metric in the σ -model frame is diagonal,

$$ds_{(\sigma)}^2 = -(dX^0)^2 + (dX^1)^2 + \cot^2 X^1 (dX^2)^2 + \tan^2 X^0 (dX^3)^2. \quad (3.38)$$

It describes a suitable real form of the direct product of two 2-dim black-hole cosets. This background was used in [14] to obtain the complete Nappi-Witten solution by $O(2, 2)$ transformations. If $(1 - \sin \theta)/\cos \theta$ takes only integer values, the transformation is in $O(2, 2; \mathbb{Z})$ and the underlying backgrounds are equivalent as exact conformal field theories. It is interesting to note that the transformation that provides the $(1, 1)$ soliton dressing already contains in it the corresponding $O(2, 2)$ group elements; but it also contains much more that allow for a flat space derivation of these CFT backgrounds.

(ii) The coset $SL(2)/SO(1, 1) \times SO(1, 1)^2$:

Following the same construction as above we will now specify other points in the moduli space of soliton solutions (3.17)–(3.19) that lead to the semi-classical geometry of the $SL(2)/SO(1, 1) \times SO(1, 1)^2$ coset.

We choose $C_1 = 0$ for the metric sector, thus arriving at the same expression (3.24) as before, while for the axion-dilaton sector we let $C'_2 = 0$. In this case we find

$$e^{-2\Phi} = C'_0 - \beta + \sqrt{(C'_0 - \beta)^2 - \alpha^2} , \quad (3.39)$$

$$b = 0 . \quad (3.40)$$

We furthermore let

$$C_0 = C'_0 , \quad (3.41)$$

and introduce the coordinate transformation

$$\alpha = \frac{1}{2}e^{2X^1} \sinh 2X^0 , \quad \beta = C_0 - \frac{1}{2}e^{2X^1} \cosh 2X^0 , \quad (3.42)$$

which clearly has $(C_0 - \beta)^2 \geq \alpha^2$ as required for reality. We find in this parametrization

$$e^{-2\Phi} = e^{2X^1} \cosh^2 X^0 , \quad (3.43)$$

while the antisymmetric tensor field is zero and the metric is diagonal. In the σ -model frame, also setting $w = X^2$ and $z = X^3$, the metric assumes the form

$$ds^2_{(\sigma)} = -(dX^0)^2 + (dX^1)^2 + \tanh^2 X^0 (dX^2)^2 + (dX^3)^2 , \quad (3.44)$$

and the resulting background coincides with the geometry of the coset $SL(2)/SO(1, 1) \times SO(1, 1)^2$ as it was advertized.

The worm-hole background will be discussed separately in section 5 using 2-dim solitons in Euclidean space.

4 Black-holes as 2-dim (2, 0) solitons

In this section we briefly review for completeness the interpretation of ordinary 4-dim black-holes as 2-dim (2, 0) solitons, filling up some of the intermediate steps of the calculation [19] as well. We use as starting point the flat space metric in polar coordinates,

$$ds^2 = d\alpha^2 + d\beta^2 + \alpha^2 d\varphi^2 - d\tau^2 , \quad (4.1)$$

for which the matrix Ψ of the linearized system (2.10) for stationary axisymmetric metrics is

$$\Psi_0(l) = \begin{pmatrix} \alpha^2 - 2\beta l - l^2 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.2)$$

We point out a few differences between this case and the cosmological setting of the two previous sections. Here, the 2-dim space (α, β) has Euclidean signature, while the σ -model g is Lorentzian. As a result, one has to take into account various sign changes in order to adopt the general soliton construction to stationary axisymmetric metrics; in particular (2.18) changes to

$$\mu_k^2 + 2(\beta - C_0^{(k)})\mu_k - \alpha^2 = 0, \quad (4.3)$$

and the factor $\mu_k\mu_l - \alpha^2$ in (2.23) changes to $\mu_k\mu_l + \alpha^2$. Analogous changes have to be introduced in the linearized system of equations (2.10)–(2.12) and the differential equations (2.19) for the poles μ_k . Also, the n -soliton transformation of a seed background g_0 yields $\det g = (-1)^n \alpha^{2n+2} \mu_1^{-2} \mu_2^{-2} \cdots \mu_n^{-2}$ and the normalization (2.25) is the same as before for n even. If n is odd, however, the signature of the soliton metric g changes sign to $++$, which is not acceptable. For this reason the simplest physical soliton to construct is the double soliton solution on flat space (4.1).

The present version of the formalism will also become relevant in the next section, while considering Euclidean gravitational solutions of the string background equations. In that case the axion-dilaton system corresponds to a Lorentzian σ -model λ , and the explicit construction of its solitons will require the modifications we are considering here.

After some calculation we find that the general $(2, 0)$ soliton on the purely gravitational background (4.1) has metric components

$$g_{\tau\varphi} = 2(\mu_1 - \mu_2)(\alpha^2 + \mu_1\mu_2)\frac{N_1}{D}, \quad g_{\tau\tau} = -4\mu_1\mu_2\frac{N_2}{D}, \quad (4.4)$$

where

$$N_1 = A\Sigma(\alpha^2 + \mu_1^2)(\alpha^2 + \mu_2^2) + AM\alpha^2(\mu_1^2 - \mu_2^2) - B\Sigma(\alpha^4 - \mu_1^2\mu_2^2), \quad (4.5)$$

$$N_2 = \Sigma^2(\alpha^2 + \mu_1\mu_2)^2 - A^2\alpha^2(\mu_1 - \mu_2)^2, \quad (4.6)$$

$$D = ((\Sigma + M)\mu_1 + (\Sigma - M)\mu_2)^2(\alpha^2 + \mu_1\mu_2)^2 + \left((A - B)\alpha^2 - (A + B)\mu_1\mu_2\right)^2(\mu_1 - \mu_2)^2, \quad (4.7)$$

while $g_{\varphi\varphi}$ is determined by the condition $\det g = -\alpha^2$. Also the corresponding conformal factor turns out to be

$$f = \frac{\mu_1\mu_2}{4(\mu_1 - \mu_2)^2(\alpha^2 + \mu_1^2)(\alpha^2 + \mu_2^2)(\alpha^2 + \mu_1\mu_2)^2} D, \quad (4.8)$$

whereas $f_0 = 1$. In the above expressions the parameters A , B , M and Σ are the special combinations of the $C^{(k)}$ moduli,

$$\begin{aligned} C_1^{(1)} C_2^{(2)} &= (\Sigma - M) C_0^{(1)} , & C_1^{(1)} C_1^{(2)} &= 2(A - B) C_0^{(1)} C_0^{(2)} , \\ C_2^{(1)} C_1^{(2)} &= -(\Sigma + M) C_0^{(2)} , & C_2^{(1)} C_2^{(2)} &= \frac{1}{2}(A + B) , \end{aligned} \quad (4.9)$$

which clearly satisfy the condition

$$\Sigma^2 = M^2 - A^2 + B^2 . \quad (4.10)$$

Also, using an appropriate shift of β we may fix,

$$\Sigma = \frac{1}{2}(C_0^{(1)} - C_0^{(2)}) , \quad Z = \frac{1}{2}(C_0^{(1)} + C_0^{(2)}) . \quad (4.11)$$

We introduce now the change of variables

$$\alpha = \sqrt{(r - M)^2 - \Sigma^2} \sin \theta , \quad \beta - Z = (r - M) \cos \theta , \quad (4.12)$$

and substitute for μ_1 and μ_2 . Provided that we choose the solutions μ_{k+} of (4.3), we obtain

$$\mu_1 = 2(r - M + \Sigma) \sin^2 \frac{\theta}{2} , \quad \mu_2 = 2(r - M - \Sigma) \sin^2 \frac{\theta}{2} . \quad (4.13)$$

Hence, the 2-soliton solution depends only on three moduli A , B and M , while Σ is fixed by (4.10) and Z does not appear anywhere. It is also useful to introduce the change of variable

$$t = -\tau + 2A\varphi \quad (4.14)$$

and identify t with the time coordinate. Then, substituting in (4.4)–(4.8) we may compute the explicit form of the 2-soliton metric in the coordinates (r, θ, φ, t) .

The special case $A = B = 0$ is the simplest, since the resulting 2-soliton metric is diagonal,

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{r}{r - 2M} dr^2 - \frac{r - 2M}{r} dt^2 , \quad (4.15)$$

and coincides with the Schwartzchild metric. In the more general situation we obtain the Kerr metric with mass parameter M , rotation parameter A and NUT parameter B that describes the behaviour of the 4-dim metric at infinity. The result of the 2-soliton construction precisely yields the complete Kerr metric in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 &= \frac{1}{r^2 + (B - A \cos \theta)^2} \left(-[(r - M)^2 - \Sigma^2 - A^2 \sin^2 \theta] dt^2 + \right. \\ &\quad \left[\sin^2 \theta (r^2 + A^2 + B^2)^2 - ((r - M)^2 - \Sigma^2) (2B \cos \theta + A \sin^2 \theta)^2 \right] d\varphi^2 + \\ &\quad 4 \left[B \cos \theta ((r - M)^2 - \Sigma^2) - A \sin^2 \theta (Mr + B^2) \right] dt d\varphi \Big) + \\ &\quad (r^2 + (B - A \cos \theta)^2) \left(d\theta^2 + \frac{1}{(r - M)^2 - \Sigma^2} dr^2 \right) . \end{aligned} \quad (4.16)$$

We note finally that the Lorentzian analogue of the self-dual Taub-NUT metric corresponds to the limit $\Sigma = 0 = A$ and $M = \pm iB$. This limit, however, is somewhat peculiar in the 2-soliton sector because all three quantities N_1 , N_2 , D become zero, and the two real poles (4.13) coincide.

5 Euclidean (0, 2) solitons

We describe now how axionic instanton solutions of the Euclidean string background equations can be accommodated into the present scheme. More specifically we study the moduli space of (0, 2) 2-dim solitons and determine the specific choice of parameters that give rise to the worm-hole (and other related backgrounds) as solitons on flat space. It is true that these backgrounds were originally introduced as soliton solutions of the 10-dim heterotic string theory (see for instance [14] and references therein), but their description as 2-dim solitons of the reduced β -function equations using inverse scattering methods appears to be new.

We consider the general form of the (0, 2) soliton solutions starting from 4-dim flat Euclidean space. It will be convenient for later use to consider as seed metric

$$ds^2 = \frac{1}{2\sqrt{\alpha^2 + \beta^2}}(d\alpha^2 + d\beta^2) + (\sqrt{\alpha^2 + \beta^2} - \beta)d\psi^2 + (\sqrt{\alpha^2 + \beta^2} + \beta)d\tau^2 \quad (5.1)$$

with $\det g = \alpha^2$, instead of the Euclidean version of (4.1). However, the explicit form of the metric background will be used only after completing the soliton construction in the axion-dilaton sector. The seed Φ and b are zero, and (5.1) is flat as can be seen using the transformation

$$\alpha = \frac{1}{2}e^{2\rho} \sin 2\varphi, \quad \beta = \frac{1}{2}e^{2\rho} \cos 2\varphi, \quad (5.2)$$

or introducing $z = \exp(\rho + i\tau) \cos \varphi$, $w = \exp(\rho + i\psi) \sin \varphi$ that yields $ds^2 = dzd\bar{z} + dwd\bar{w}$.

The axion-dilaton sector has Lorentzian signature (we should replace $b^2 + e^{-4\Phi}$ with $b^2 - e^{-4\Phi}$ in (2.1), since the relevant conjugate pair of field variables is now $S_{\pm} = b \pm e^{-2\Phi}$ instead of $b \pm ie^{-2\Phi}$), and therefore the (0, 2) soliton calculation is analogous to (2, 0) metric solitons of stationary axisymmetric gravity. There is a difference with the analysis of the previous section, however, in that our normalized axion-dilaton seed matrix $\alpha\lambda_0 = \text{diag}(\alpha, -\alpha)$ is “isotropic”, while $g_0 = \text{diag}(\alpha^2, -1)$ in (4.1). It is therefore appropriate in the present case to consider

$$\Psi_0(l) = \sqrt{\alpha^2 - 2\beta l - l^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.3)$$

Explicit calculation shows that the (0, 2) axion-dilaton soliton fields are

$$e^{-2\Phi} = 2\mu_1\mu_2 \frac{N'_1}{D'}, \quad b = (\mu_1 - \mu_2)(\alpha^2 + \mu_1\mu_2) \frac{N'_2}{D'}, \quad (5.4)$$

where

$$N'_1 = B^2 \alpha^2 (\mu_1 - \mu_2)^2 + C^2 (\alpha^2 + \mu_1 \mu_2)^2 , \quad (5.5)$$

$$N'_2 = AC(\mu_1 + \mu_2)(\alpha^2 + \mu_1 \mu_2) + BD(\mu_1 - \mu_2)(\alpha^2 - \mu_1 \mu_2) , \quad (5.6)$$

$$D' = C^2(\mu_1^2 + \mu_2^2)(\alpha^2 + \mu_1 \mu_2)^2 - B^2(\mu_1 - \mu_2)^2(\alpha^4 + \mu_1^2 \mu_2^2) - (\mu_1 - \mu_2) \cdot (\alpha^2 + \mu_1 \mu_2) \left(AB(\mu_1 - \mu_2)(\alpha^2 - \mu_1 \mu_2) + CD(\mu_1 + \mu_2)(\alpha^2 + \mu_1 \mu_2) \right) . \quad (5.7)$$

Here we have used for convenience the notation

$$\begin{aligned} C_1^{(1)} C_1^{(2)} &= \frac{1}{2}(A + B) , & C_1^{(1)} C_2^{(2)} &= -\frac{1}{2}(C + D) , \\ C_2^{(1)} C_2^{(2)} &= \frac{1}{2}(A - B) , & C_1^{(2)} C_2^{(1)} &= \frac{1}{2}(C - D) , \end{aligned} \quad (5.8)$$

which satisfy the relation

$$C^2 = D^2 - A^2 + B^2 . \quad (5.9)$$

So, the soliton moduli depends on four parameters in this case, A , B , C and one of the $C_0^{(k)}$, since the other one can be absorbed by shifting β .

Searching for an axionic instanton solution in this moduli space we notice that if

$$A = \pm D ; \quad B^2 = C^2 \quad (5.10)$$

we obtain

$$e^{-2\Phi} \mp b = 1 . \quad (5.11)$$

One can also prove relatively easy that there are no other axionic instanton solutions in this sector. Moreover, from all soliton constructions we have considered so far, only this example exhibits axionic instanton solutions.

Next, we determine the explicit form of the solution choosing $A = D$ and $B = C$ for concreteness. Substituting in (5.4) we obtain

$$e^{2\Phi} = \frac{D'}{2\mu_1 \mu_2 N'_1} = 1 - \frac{A}{B} \frac{(\mu_1 - \mu_2)(\alpha^2 + \mu_1 \mu_2)}{\mu_2(\alpha^2 + \mu_1^2)} . \quad (5.12)$$

Further manipulation using (4.3) yields

$$e^{2\Phi} = 1 - 2 \frac{A}{B} (C_0^{(1)} - C_0^{(2)}) \frac{1}{\mu_1 + \frac{\alpha^2}{\mu_1}} \quad (5.13)$$

and so the dilaton depends only on μ_1 . Choosing the solution μ_+ and taking into account that $\mu_- = -\alpha^2/\mu_+$ in this case, we finally arrive at the result

$$e^{2\Phi} = 1 + \frac{M}{\sqrt{(C_0^{(1)} - \beta)^2 + \alpha^2}} ; \quad M = (C_0^{(2)} - C_0^{(1)}) \frac{A}{B} . \quad (5.14)$$

$C_0^{(1)}$ can be absorbed by shifting β , and hence for all purposes it may be set equal to zero.

We also note for completeness that for $A = D$, but $B = -C$, the dilaton depends only on μ_2 and the result turns out to be essentially the same up to an interchange of $C_0^{(1)}$ and $C_0^{(2)}$. Similar remarks apply to the axionic anti-instanton case $A = -D$ with $B = \pm C$.

The solution we have obtained in this fashion can be put together with the flat space metric (5.1) to yield in the Einstein frame the 4-dim string background

$$ds^2 = e^{2\rho}(d\rho^2 + d\varphi^2 + \sin^2\varphi d\psi^2 + \cos^2\varphi d\tau^2), \quad (5.15)$$

$$e^{2\Phi} = 1 + 2Me^{-2\rho}, \quad b = e^{-2\Phi} + \text{const.} \quad (5.16)$$

We have used the change of coordinates (5.2) and the fact that for axionic instantons there is no contribution to the conformal factor f coming from the λ sector (see eqs. (2.8), (2.9)). This configuration is actually known as the worm-hole solution. A related version of it, which represents only the throat of the worm-hole and coincides with the semi-classical geometry of the $SU(2) \times U(1)$ WZW model, consists of a dilaton field with different boundary condition in that the constant term in (5.16) is missing, but with the same Einstein metric. These two models are related to each other by an $SL(2, R)$ transformation (the continuous counterpart of S-duality), which keeps the axionic instanton condition invariant, say S_- constant, and simply shifts $e^{2\Phi}$ by a constant.

We finally point out that higher solitonic excitations of the axion-dilaton sector might be interesting to consider in this case. We have determined the solution of the linearized system for the λ sector of the semi-classical background of the model $SU(2) \times U(1)$,

$$\Psi_{\text{wh}}(l) = \begin{pmatrix} \frac{\alpha^2 - \beta l}{2\sqrt{(\alpha^2 + \beta^2)(\alpha^2 - 2\beta l - l^2)}} & \sqrt{\alpha^2 - 2\beta l - l^2} \\ \sqrt{\alpha^2 - 2\beta l - l^2} & 0 \end{pmatrix}. \quad (5.17)$$

However, we are not in a position at this moment to give a good space-time interpretation to its multi-soliton excitations. It might turn out an interesting geometrical problem and we hope to return to it elsewhere.

6 Conclusions and discussion

In this paper we have considered solitonic solutions of the 2-dim reduced β -function equations for gravitational string backgrounds with axion and dilaton fields. We found that many known solutions that admit an exact conformal field theory description arise as simple solitonic excitations of flat space, or its T-dual face, depending on the particular examples. These backgrounds include cosmological solutions, as well as 4-dim black-holes and worm-holes. It should be also interesting to consider further generalizations to colliding gravitational plane wave solutions of string theory, thus extending the results of [20] to strings.

A key point in implementing our construction is the non-trivial interplay between the coordinate and the solution generating (Geroch-type) transformations. These two kind of transformations do not commute with each other, and we should always use a coordinate system in the seed background with non-constant $\det g$. The Geroch transformations leave $\det g$ invariant, and only after the solitonic dressing is performed we may transform the new background into any other suitable coordinate system. Note that the action of the Geroch group would be trivial if we were starting from flat space in Cartesian coordinates because there are no other physical solutions of the Ernst σ -model with $\det g = \pm 1$ that are also compatible with the equations for the conformal factor f . Actually, the conformal factor has a very special role incorporating the effects of 2-dim gravity in the reduced form of the string background equations.

For certain discrete values of the soliton moduli the underlying string backgrounds are equivalent as exact conformal field theories, as it was demonstrated explicitly for the general one-parameter form of the Nappi-Witten universe. This happens because the corresponding $O(2, 2; Z)$ T-duality transformations [17] act on the soliton moduli as described. Since the soliton transformations provide specific elements of the string Geroch group, i.e. the current group $\hat{O}(2, 2)$ in the simplest case under discussion, and they also contain S transformations and their intertwining with T, our description suggests that other discrete remnants of this infinite dimensional group could act as U-dualities in string compactifications to two dimensions [4, 10].

Further progress in this direction certainly requires extending the formalism to include gauge fields as well. For example, it will be interesting to describe 4-dim black-holes with electric charge Q as 2-dim solitons of an Ernst-type $SU(2, 1)$ σ -model. The points in the soliton moduli space that describe extremal black-holes are special in that a solitonic interpretation of the resulting configuration exists in four as well as in two (reduced) dimensions. This should be ultimately extended to $O(8, 24)$ σ -models, which are applicable to heterotic string compactifications to two dimensions [10, 11]. One should also try to find in this context the necessary conditions on the Killing isometries so that the solitonic nature of a given configuration is preserved or attained under reduction. It seems that the relevant distinction in the examples we have considered so far is provided by the translational versus the rotational character of the corresponding Killing vector fields (using the same terminology as in [3]). For Euclidean black-holes, for example, $\partial/\partial t$ is rotational unless $M = B$ (self-dual Taub-NUT limit with $A = 0$ in the vacuum case) or $M = Q$ (extremal non-rotating solution of the electrovacuum equations), in which cases we find that $\partial/\partial t$ becomes translational. These two examples demonstrate clearly that the reduction with respect to rotational isometries can give rise to 2-dim solitonic configurations, in the sense described above, even if the 4-dim configurations are not so. A characteristic feature of rotational isometries in supersymmetric backgrounds is that the fermions depend on the Killing coordinates, while the bosonic fields do not, and hence it is like having a coordinate dependent compactification.

The soliton solutions of the 4-dim theory (BPS states) are quantum mechanically stable having manifest space-time supersymmetry. On the other hand, the 2-dim solitonic

interpretation of a given configuration was only used here in the context of the inverse scattering method with no reference to supersymmetry. Hence, it arises a natural question to find the special properties of these 2-dim solitons with respect to 2-dim reductions of the space-time supersymmetry algebra. To the best of our knowledge the only helpful results that exist in the literature are contained in [22], where the 2-dim reduction of maximal supergravity (and some of its consistent truncations) were described in terms of Lax pairs and integrable structures. We briefly mention that the local supersymmetry transformations can be bosonized in two dimensions and they can be included as Kac-Moody variations into the corresponding infinite dimensional hidden symmetry (Geroch-type) transformations. Also, the variation of the conformal factor that accounts for the 2-dim gravitational effects in this case can be included systematically using central extensions of the associated current groups [23]. Further work is certainly required in this direction, putting together in a more constructive way the vast variety of soliton solutions obtained by the inverse scattering methods with the supersymmetric configurations of the 2-dim reduced supergravity. It is conceivable that 11-dim supergravity will be singled out in this line of thought, according to earlier expectations by Nicolai [22].

Summarizing, the 2-dim reduced sector of string theory is quite rich in symmetry due to the integrability structure of the lowest order effective theory. Better understanding of its soliton solutions are worthy in the light of the recent developments in non-perturbative string theory, in order to establish and explore the meaning of infinitely many U-dualities in the spectrum. This particular sector of string theory, where many more connections are expected to exist than in other sectors, could also help in the long run to expose better the right structures that are needed for the ultimate formulation of string theory. Could it be an exactly solvable sector? We hope to return to it elsewhere.

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